Orientability, De Rham Cohomology

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Exercise 1 (Orientability of the sphere)**.** Prove that $\mathbb{S}^n$ is orientable.

Exercise 2 (Orientability of the projective space)**.**

1. Show that if $\widehat{M}$ is orientable and connected, and $M = \widehat{M}/\Gamma$ is the quotient of M by a group of diffeomorphisms (acting properly discontinuously), one of which reverses the orientation of $\widehat{M}$, then M is non-orientable.
2. Deduce that $\mathbb{RP}^{2n}$ is non-orientable.
3. Show that $\mathbb{RR}^{2n+1}$ is orientable.

Exercise 3 (Homotopy equivalence)**.** A smooth map $f : M \to N$ is a *homotopy equivalence* if there exists a smooth map $g : N \to M$ such that $f \circ g$ is homotopic to id_N and $g \circ f$ is homotopic to id_M. In this case, we say that M and N have the same homotopy type.

1. Prove that if M and N have the same homotopy type, then $H^*(M) = H^*(N)$.
2. Prove that $H^*(M \times \mathbb{R}^d) = H^*(M)$.
3. Prove that $H^*(\mathbb{R}^n \setminus \{0\}) = H^*(\mathbb{S}^{n-1})$.

Exercise 4 (Orientability)**.**

1. Show that any parallelizable manifold is orientable.
2. Show that a product of two orientable manifolds is itself orientable.
3. Show that the tangent bundle of a manifold is an orientable manifold.
4. Give an explicit volume form on the cotangent bundle.

Exercise 5 (Divergence and curl of a vector field (continued))**.**

1. Let $(E, \langle \cdot\,, \cdot \rangle)$ be an oriented Euclidean space of dimension 3. Show that any 1-form of E can be written as $\langle X\,, \cdot \rangle$ for some vector field X on E. Deduce that for every vector field X such that $\operatorname{div}(X) = 0$, there exists a vector field A of E such that $X = \operatorname{curl} A$.
2. Show that for every vector field X such that $\operatorname{curl} X = 0$, there exists a smooth function $f : E \to \mathbb{R}$ such that $X = \nabla f$.
3. Let $D := \{(0, 0)\} \times \mathbb{R} \subset \mathbb{R}^3$. Show that $H^*(\mathbb{R}^3 \setminus D) \simeq H^*(\mathbb{R}^2 \setminus 0)$.
4. We recall that $H^1(\mathbb{R}^2 \setminus 0)$ is generated by the angular 1-form $[\mathrm{d}\theta]$ and $H^2(\mathbb{R}^2 \setminus 0) = 0$. Let R be the vector field of $\mathbb{R}^3 \setminus D$ defined by
$$R(r\cos\theta, r\sin\theta, z) := -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}, \quad \forall r > 0, \forall\theta \in \mathbb{R}, \forall z \in \mathbb{R}.$$
Show that $H^1(\mathbb{R}^3 \setminus D)$ is generated by $\langle R\,, \cdot \rangle$.
5. Deduce that for every vector field X of $\mathbb{R}^3 \setminus D$ such that $\operatorname{curl} X = 0$, there exists $\lambda \in \mathbb{R}$ and a smooth $f : \mathbb{R}^3 \setminus D \to \mathbb{R}$ such that $X = \lambda R + \nabla f$. What about those vector fields satisfying $\operatorname{div} X = 0$?

6. Let α be the 1-form $\langle X, \cdot \rangle$. Prove that

$$\lambda = \frac{1}{2\pi} \int_\gamma \alpha,$$

where $\gamma(t) = (\cos(t), \sin(t), 0)$.